Artificial Intelligence

LECTURE 5

Reasoning

***modus ponens*:**

A
IF A, THEN B

B

The validity of ***A*** and ***IF A, THEN B***, allows the derivation of *B*.

Classical Set (Crisp Set)

A classical set (crisp set) – a collection of distinct well-defined objects.

$a \in A$ - an element a to the set A

$b \notin A$ - an element b does not belong to the set A

<u>The characteristic function:</u>

$\phi_A(x) = 1$ - if an element x belongs to the set A

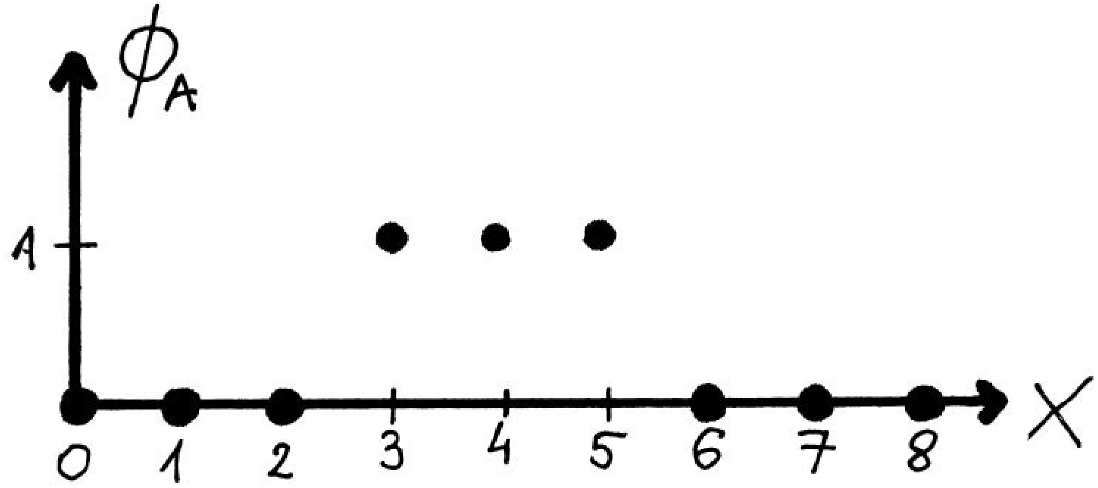
$\phi_A(x) = 0$ - if an element x does not belong to the set A

Classical Set (Crisp Set)

Example

U – a universe (the elements of interest): a set of numbers {0,1,2,3,4,5,6,7,8}
A – a set of numbers close to 4, e.g. A={3,4,5}

$$3,4,5 \in A$$
$$0,1,2,6,7,8 \notin A$$



Fuzzy Set

A fuzzy set – a set whose elements have degrees of membership.

The characteristic function (membership function):

$\mu_A(x)=1$	- if an element x belongs to the set A
$0<\mu_A(x)<1$	- if an element x belongs to the set A to a certain degree
$\mu_A(x)=0$	- if an element x does not belong A

Fuzzy Set

$$A=\{(x,\mu_A(x)):x\in \boldsymbol{X}\}, \quad where\ \mu_A\in[0,1]$$

$\boldsymbol{X}$ – a universe

$$A=\frac{\mu_A(x_1)}{x_1}+\frac{\mu_A(x_2)}{x_2}+\ldots+\frac{\mu_A(x_k)}{x_k}$$

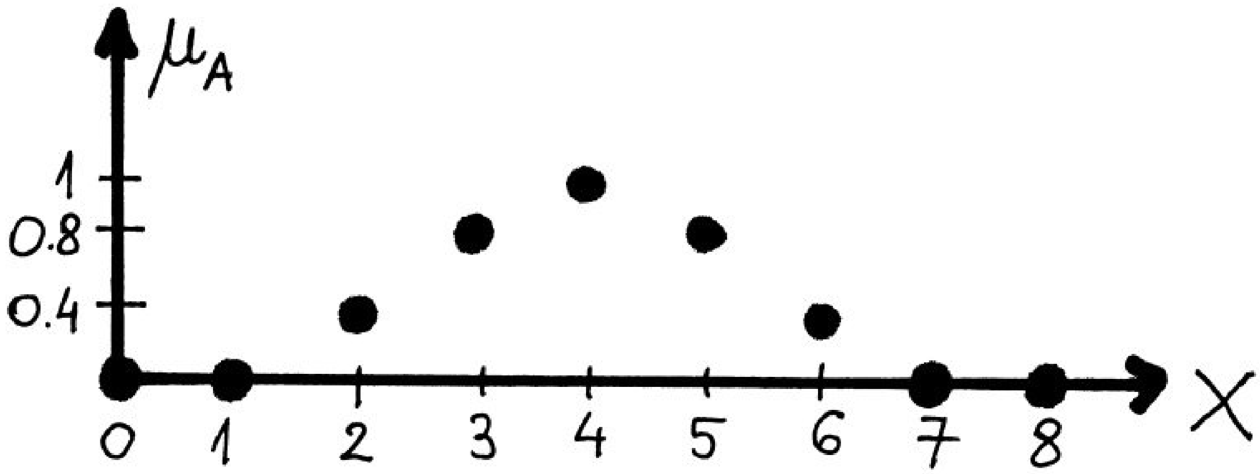
$$x_{1,}x_{2,}\ldots,x_k\in \boldsymbol{X}$$

Fuzzy Set

Example

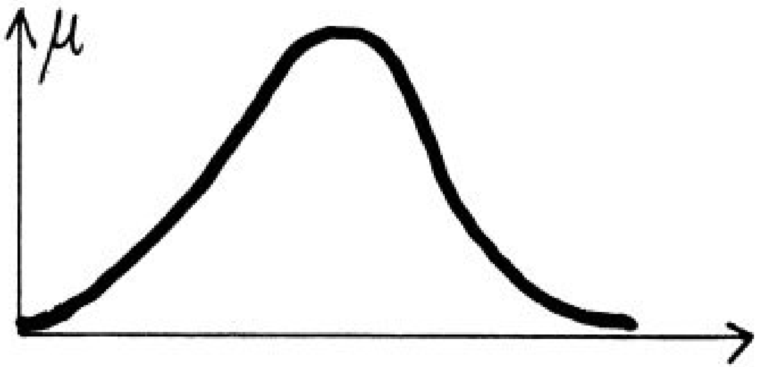
U – a universe (the elements of interest): a set of numbers {0,1,2,3,4,5,6,7,8}
A – a set of numbers close to 4

$$A = \frac{0}{0} + \frac{0}{1} + \frac{0.4}{2} + \frac{0.8}{3} + \frac{1}{4} + \frac{0.8}{5} + \frac{0.4}{6} + \frac{0}{7} + \frac{0}{8}$$

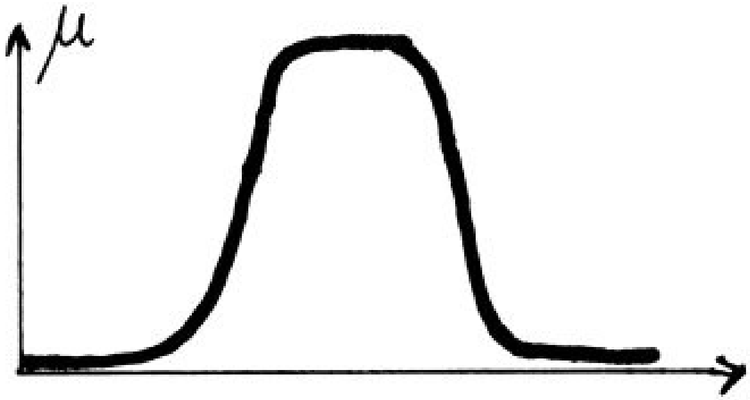


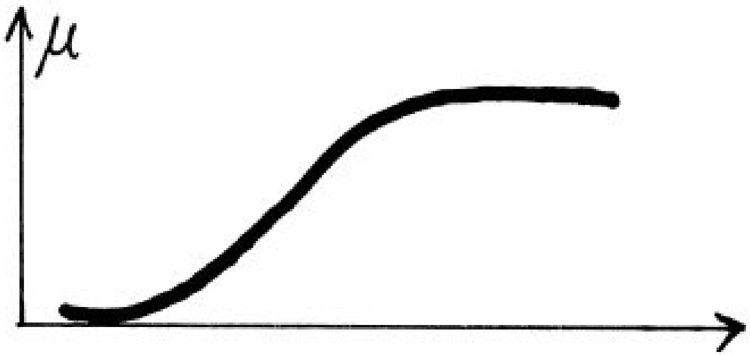
Membership Functions

The Gaussian function



The π function

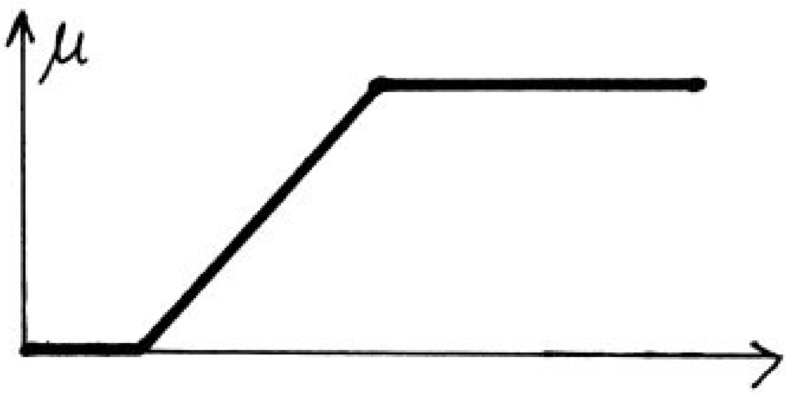




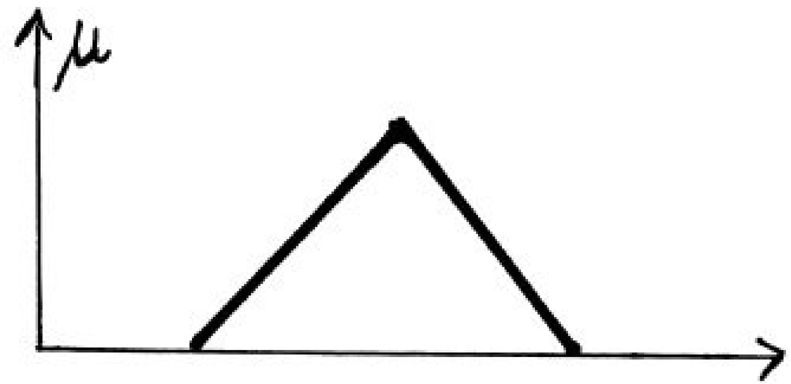
The sigmoid function

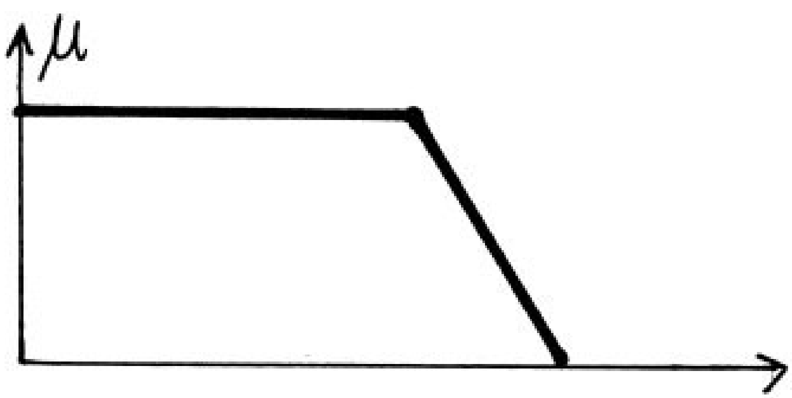
Membership Functions

The Γ function

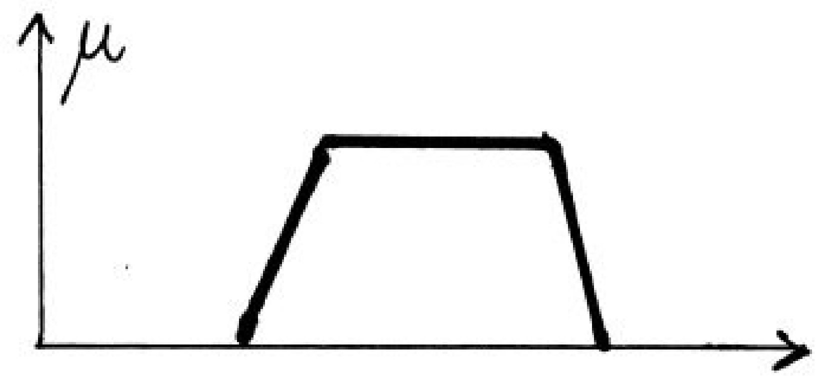


The Λ function





The L function



The Π function

Operations on Fuzzy Sets

A, *B* – fuzzy sets in ***X***

Intersection:

$$\mu_{A\cap B}(x)=min(\mu_A(x),\mu_B(x)) \quad for\ each\ x\in \boldsymbol{X}$$

Union:

$$\mu_{A\cup B}(x)=max(\mu_A(x),\mu_B(x)) \quad for\ each\ x\in \boldsymbol{X}$$

Complement:

$$\mu_{\neg A}(x)=1-\mu_A(x) \quad for\ each\ x\in \boldsymbol{X}$$

T-Norms

A, *B* – fuzzy sets in ***X***

Intersection:

$$\mu_{A \cap B}(x) = T(\mu_A(x), \mu_B(x)) \quad \text{\textit{for each}}\ x \in \boldsymbol{X}$$

T – *t*-norm

Union:

$$\mu_{A \cup B}(x) = S(\mu_A(x), \mu_B(x)) \quad \text{\textit{for each}}\ x \in \boldsymbol{X}$$

S – *s*-norm

T-Norms

t-norm

$$T:[0,1]\times[0,1]\rightarrow[0,1]$$

$$T(a,c)\leqslant T(b,d) \quad for \quad a\leqslant b, c\leqslant d$$

$$T(a,b)=T(b,a)$$

$$T(T(a,b),c)=T(a,T(b,c))$$

$$T(a,1)=a$$

$$T(a,0)=0$$

$$a,b,c,d\in[0,1]$$

S-Norms

s-norm

$$S:[0,1]\times[0,1]\rightarrow[0,1]$$

$$S(a,c)\leqslant S(b,d) \quad \mathit{for} \quad a\leqslant b,\ c\leqslant d$$
$$S(a,b)=S(b,a)$$
$$S(S(a,b),c)=S(a,S(b,c))$$
$$S(a,0)=a$$
$$S(a,1)=1$$

$$a,b,c,d\in[0,1]$$

T-Norms and S-norms

The examples of *t*-norms

$$T(a,b)=min(a,b)$$ - minimum

$$T(a,b)=a\cdot b$$ - algebraic product

$$T(a,b)=max(0,a+b-1)$$ - Lukasiewicz *t*-norm

The examples of *s*-norms

$$S(a,b)=max(a,b)$$ - maximum

$$S(a,b)=a+b-a\cdot b$$ - probabilistic product

$$S(a,b)=min(a+b,1)$$ - Lukasiewicz *s*-norm

Fuzzy Relations

X, Y – crisp sets, R – a fuzzy relation in X x Y

$$R=\{((x,y),\mu_R(x,y)): x\in X, y\in Y\}, \quad where\, \mu_R(x,y)\in[0,1]$$

$$\begin{array}{ccccc}
 & y_1 & y_2 & \dots & y_m \\
x_1 & \mu_R(x_{1,} y_1) & \mu_R(x_{1,} y_2) & \dots & \mu_R(x_{1,} y_m) \\
x_2 & \mu_R(x_{2,} y_1) & \mu_R(x_{2,} y_2) & \dots & \mu_R(x_{2,} y_m) \\
\dots & \dots & \dots & \dots & \dots \\
x_n & \mu_R(x_n, y_1) & \mu_R(x_n, y_2) & \dots & \mu_R(x_n, y_m)
\end{array}$$

$$x_{1,} x_{2,} \dots, x_n \in X, y_{1,} y_{2,} \dots, y_m \in Y$$

Fuzzy Relations

A – a fuzzy set in X, B – a fuzzy set in Y, R – a fuzzy relation in X x Y

$$R = \{((x, y), min(\mu_A(x), \mu_B(y))) : x \in X, y \in Y\}$$

Fuzzy Reasoning

<u>Fuzzy *modus ponens*:</u>

A' ← **a fuzzy set in *X***

IF A, THEN B ← **a fuzzy implication represented by a fuzzy relation *R* in *X* x *Y***

B'=? ← **a fuzzy set *Y***

$$B' = A' \circ R$$

$$\mu_{B'} = \max_{x \in X}(T(\mu_{A'}(x), \mu_R(x, y)))$$

Fuzzy Reasoning

Example

$X=\{1,2,3\}, \quad Y=\{1,2,3,4\}$

$$LOW=\frac{1}{1}+\frac{0.7}{2}+\frac{0.3}{3} \qquad HIGH=\frac{0.2}{1}+\frac{0.5}{2}+\frac{0.8}{3}+\frac{1}{4}$$

$$R=LOW\times HIGH$$

X Y	1	2	3	4
1	0.2	0.5	0.8	1
2	0.2	0.5	0.7	0.7
3	0.2	0.3	0.3	0.3

Fuzzy Reasoning

Example

$$MEDIUM = \frac{0.5}{1} + \frac{1}{2} + \frac{0.5}{3}$$

$$MEDIUM \circ R = \max_{x \in \{1,2,3\}} \left(T\left(\mu_{MEDIUM}(x), \mu_R(x, y) \right) \right) =$$

$$= \frac{0.2}{1} + \frac{0.5}{2} + \frac{0.7}{3} + \frac{0.7}{4}$$

T-norm: minimum.

Introduction to Rough Sets

- Rough set theory was developed by Zdzislaw Pawlak (1926 – 2006) in the early 1980's.
- It can be used, among others, for feature selection, feature extraction, data reduction, decision rule generation, pattern extraction.

Basic Concepts of Rough Sets

- Information / Decision System
- Indiscernibility Relation
- Approximation of Sets
- Reducts
- Dependency of Attributes

Indiscernibility Relation

- The starting point of rough set theory is the *indiscernibility relation*.
- The indiscernibility relation identifies objects having the same properties.
- Objects having the same properties are indiscernible and are treated as identical or similar.

Indiscernibility Relation (*cont.*)

$S=(U, A)$ – an information system.

$B \subseteq A$

$$Ind_B(S)=\{(x, y) \in U \times U : a(x)=a(y) \text{ for each } a \in B\}$$

If a pair (x, y) *of objects belongs to* $Ind_B(S)$*, then we say that objects* x *and* y *are indiscernible with respect to attributes from the set* B.

Indiscernibility Relation - Example (cont.)

U/A	*e*	*q*	*c*	*r*	*t*
s1	*high*	*good*	*yes*	*yes*	*no*
s2	*high*	*good*	*no*	*yes*	*no*
s3	*medium*	*good*	*yes*	*yes*	*yes*
s4	*low*	*avg*	*yes*	*yes*	*yes*
s5	*low*	*good*	*yes*	*yes*	*yes*
s6	*high*	*avg*	*no*	*no*	*yes*

It is easy to see that each store has different description in terms of attributes: *e*, *q*, *c*, *r*, *t*.

All stores are discernible by employing information provided by all attributes.

Indiscernibility Relation - Example (cont.)

U/A	e	r	t
s1	*high*	*yes*	*no*
s2	*high*	*yes*	*no*
s3	*medium*	*yes*	*yes*
s4	*low*	*yes*	*yes*
s5	*low*	*yes*	*yes*
s6	*high*	*no*	*yes*

If we take only three attributes (*e*, *r*, *t*) to describe stores, then stores *s1* and *s2* are indiscernible (i.e., they cannot be distinguished). Stores *s4* and *s5* are also indiscernible.

$$B = \{e, r, t\}$$

$$Ind_B(S) = \{(s1, s1), (s2, s2), (s3, s3), (s4, s4), (s5, s5), (s6, s6), (s1, s2), (s2, s1), (s4, s5), (s5, s4)\}$$

Equivalence classes

$S=(U, A)$ – *an information system*

$B \subseteq A$ – *a given subset of attributes*

$Ind_B(S)$ – *indiscernibility relation determined by* B

An indiscernibility relation is an equivalence relation (i.e., it is reflexive, symmetric and transitive).

$B(x)$ *denotes an equivalence class of* $Ind_B(S)$ *including an object* x:

$$B(x)=\{y \in U : (x, y) \in Ind_B(S)\}$$

Equivalence classes (*cont.*)

$S=(U, A)$ – *an information system*

$B \subseteq A$ – *a given subset of attributes*

The subset B determines the partition of the set U of objects. This partition is denoted by U/B.

Equivalence classes - Example

U/A	e	r	t
s1	high	yes	no
s2	high	yes	no
s3	medium	yes	yes
s4	low	yes	yes
s5	low	yes	yes
s6	high	no	yes

$$B = \{e, r, t\}$$

$$B(s1) = B(s2) = \{s1, s2\}$$
$$B(s3) = \{s3\}$$
$$B(s4) = B(s5) = \{s4, s5\}$$
$$B(s6) = \{s6\}$$

The partition of U determined by the subset B of attributes:

$$U / B = \{\{s1, s2\}, \{s3\}, \{s4, s5\}, \{s6\}\}$$

Approximation of Sets

$S=(U, A) - an\ information\ system$

$B \subseteq A - a\ given\ subset\ of\ attributes$

$X \subseteq U - a\ given\ subset\ of\ objects\ (concept)$

$The\ B-lower\ approximation\ of\ set\ X:$

$$\underline{B}X = \{x \in U : B(x) \subseteq X\}$$

$The\ B-upper\ approximation\ of\ set\ X:$

$$\overline{B}X = \{x \in U : B(x) \cap X \neq \emptyset\}$$

Approximation of Sets (*cont.*)

$The\ \ B-positive\ \ region\ \ of\ \ \ set\ \ X:$

$$POS_B(X)=\underline{B}X$$

$The\ \ B-boundary\ \ region\ \ of\ \ \ set\ \ X:$

$$BN_B(X)=\overline{B}X-\underline{B}X$$

$The\ \ B-negative\ \ region\ \ of\ \ \ set\ \ X:$

$$NEG_B(X)=U-\underline{B}X$$

Approximation of Sets (*cont.*)

- The *B*-lower approximation of set *X* consists of all objects from *U* which can be classified with certainty as elements of set *X* having the knowledge about them represented by attributes from set *B*.
- The *B*-boundary region of set *X* consists of objects from *U* which one can classify neither to set *X* nor to set *U*-*X* having the knowledge about them represented by attributes from set *B*.

Approximation of Sets (*cont.*)

- If the boundary region of a given set (concept) X is the empty set, then we say that X is ***crisp*** (***exact***) with respect to a given set of attributes.
- Otherwise, we say that X is ***rough*** (***inexact***) with respect to a given set of attributes.

Approximation of Sets - Example

U/A	e	r	t	p
s1	high	yes	no	yes
s2	high	yes	no	no
s3	medium	yes	yes	yes
s4	low	yes	yes	yes
s5	low	yes	yes	yes
s6	high	no	yes	no

?

Question:

What are characteristic features of stores having profit (or loss) in view of available information?

Approximation of Sets – Example (*cont.*)

U/A	e	r	t	p
s1	high	yes	no	yes
s2	high	yes	no	no
s3	medium	yes	yes	yes
s4	low	yes	yes	yes
s5	low	yes	yes	yes
s6	high	no	yes	no

$$X_{profit} = \{ s1, s3, s4, s5 \}$$

$$X_{loss} = \{ s2, s6 \}$$

Approximation of Sets – Example (*cont.*)

U/A	e	r	t	p
s1	high	yes	no	yes
s2	high	yes	no	no
s3	medium	yes	yes	yes
s4	low	yes	yes	yes
s5	low	yes	yes	yes
s6	high	no	yes	no

Equivalence classes with respect to the attribute set $B=\{e,r,t\}$:

$$B(s1)=B(s2)=\{s1,s2\}$$

$$B(s3)=\{s3\}$$

$$B(s4)=B(s5)=\{s4,s5\}$$

$$B(s6)=\{s6\}$$

Approximation of Sets – Example (*cont.*)

U/A	e	r	t	p
s1	high	yes	no	yes
s2	high	yes	no	no
s3	medium	yes	yes	yes
s4	low	yes	yes	yes
s5	low	yes	yes	yes
s6	high	no	yes	no

The B-lower approximation of set X_{profit}:

$$\underline{B}X_{profit} = \{s3, s4, s5\}$$

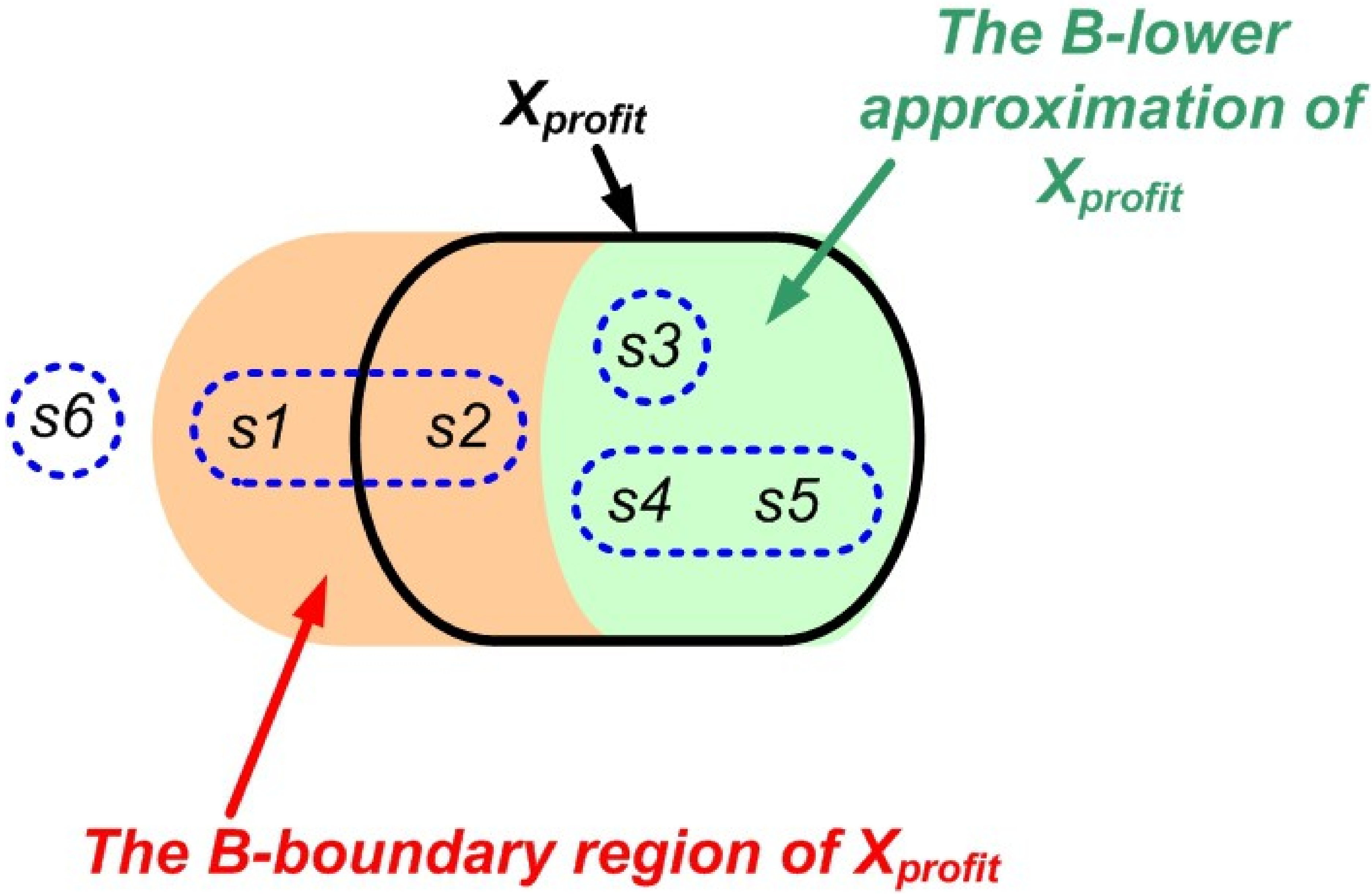
The B-upper approximation of set X_{profit}:

$$\overline{B}X_{profit} = \{s1, s2, s3, s4, s5\}$$

The B-boundary region of set X_{profit}:

$$BN_B(X_{profit}) = \overline{B}X_{profit} - \underline{B}X_{profit} = \{s1, s2\}$$

Approximation of Sets - Example (cont.)



Approximation of Sets - Example (cont.)

What are characteristic features of stores having profit (or loss) in view of available information?

Answer

- The question cannot be answered uniquely.
- We can give partial answer to this question only.
- We can say that stores $s3$, $s4$, $s5$ surely make a profit. Stores $s1$, $s2$ possibly make a profit. Story $s6$ surely has a loss.
- The set of stores making profit is rough with respect to attributes e, r and t.

Approximation of Sets (cont.)

- An equivalence relation as a basis for rough set theory seems to be a very intuitive choice, but for many applications it is not sufficient.
- Many authors proposed another relations, for example, a tolerance relation, an ordering relation and others.